PERIMETER AND AREA

Objectives:

After completing this section, you should be able to do the following:

- Calculate the area of given geometric figures.
- Calculate the perimeter of given geometric figures.
- Use the Pythagorean Theorem to find the lengths of a side of a right triangle.
- Solve word problems involving perimeter, area, and/or right triangles.

Vocabulary:

As you read, you should be looking for the following vocabulary words and their definitions:

- polygon
- perimeter
- area
- trapezoid
- parallelogram
- triangle
- rectangle
- circle
- circumference
- radius
- diameter
- legs (of a right triangle)
- hypotenuse

Formulas:

You should be looking for the following formulas as you read:

- area of a rectangle
- area of a parallelogram
- area of a trapezoid
- area of a triangle
- Heron's Formula (for area of a triangle)
- circumference of a circle
- area of a circle
- Pythagorean Theorem

We are going to start our study of geometry with two-dimensional figures. We will look at the one-dimensional distance around the figure and the two-dimensional space covered by the figure. perimeter

The *perimeter* of a shape is defined as the distance around the shape. Since we usually discuss the perimeter of polygons (closed plane figures whose sides are straight line segment), we are able to calculate perimeter by just adding up the lengths of each of the sides. When we talk about the perimeter of a circle, we call it by the special name of *circumference*. Since we don't have straight sides to add up for the circumference (perimeter) of a circle, we have a formula for calculating this. circumference

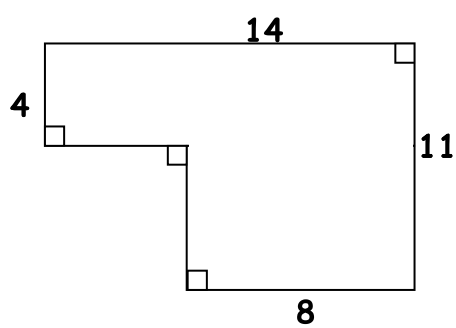
Circumference (Perimeter) of a Circle

$$C = 2\pi r$$

r = radius of the circle
π = the number that is approximated by 3.141593

Example 1:

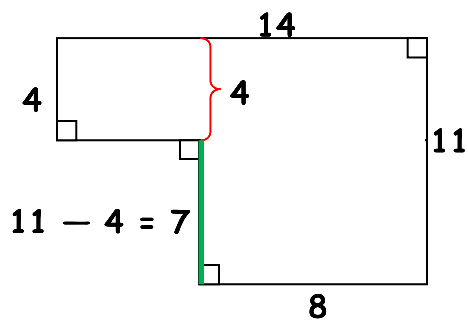
Find the perimeter of the figure below



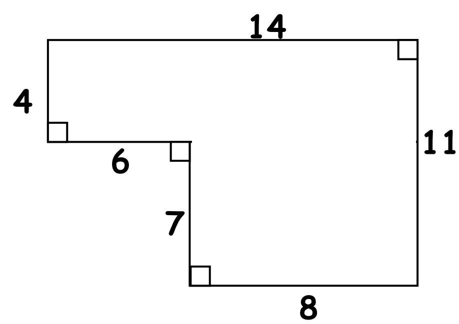
Solution:

It is tempting to just start adding of the numbers given together, but that will not give us the perimeter. The reason that it will not is that this figure has SIX sides and we are only given four numbers. We must first determine the lengths of the two sides that are not labeled before we can find the perimeter. Let's look at the figure again to find the lengths of the other sides.

Since our figure has all right angles, we are able to determine the length of the sides whose length is not currently printed. Let's start with the vertical sides. Looking at the image below, we can see that the length indicated by the red bracket is the same as the length of the vertical side whose length is 4 units. This means that we can calculate the length of the green segment by subtracting 4 from 11. This means that the green segment is 7 units.



In a similar manner, we can calculate the length of the other missing side using $14 - 8 = 6$. This gives us the lengths of all the sides as shown in the figure below.



Now that we have all the lengths of the sides, we can simply calculate the perimeter by adding the lengths together to get $4 + 14 + 11 + 8 + 7 + 6 = 50$. Since perimeters are just the lengths of lines, the perimeter would be 50 units.

area

The *area* of a shape is defined as the number of square units that cover a closed figure. For most of the shape that we will be dealing with there is a formula for calculating the area. In some cases, our shapes will be made up of more than a single shape. In calculating the area of such shapes, we can just add the area of each of the single shapes together.

We will start with the formula for the area of a rectangle. Recall that a *rectangle* is a quadrilateral with opposite sides parallel and right interior angles. rectangle

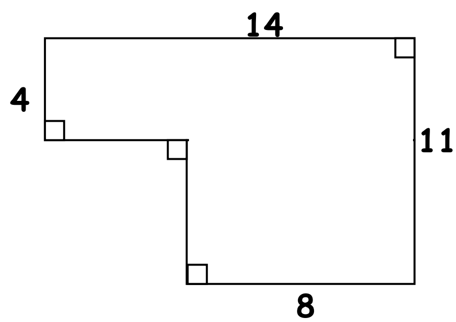
Area of a Rectangle

$$A = bh$$

b **= the base of the rectangle**
h **= the height of the rectangle**

Example 2:

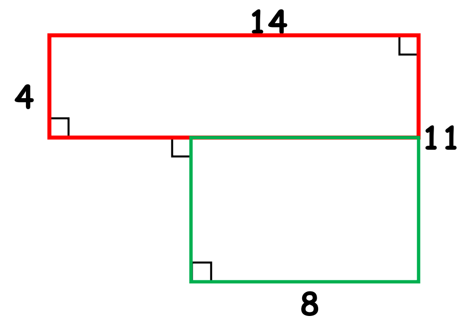
Find the area of the figure below

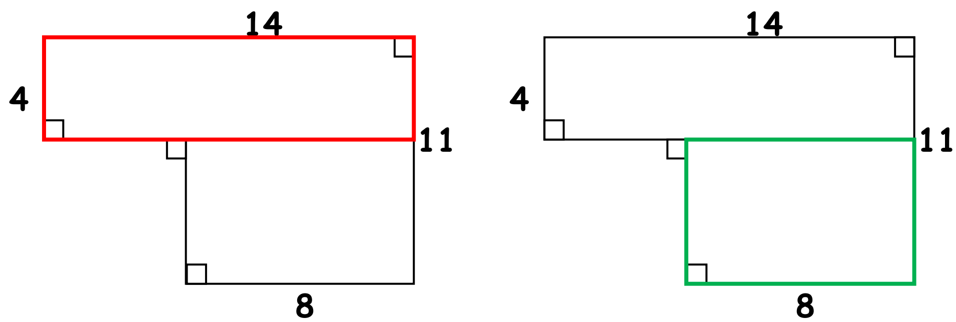


Solution:

This figure is not a single rectangle. It can, however, be broken up into two rectangles. We then will need to find the area of each of the rectangles and add them together to calculate the area of the whole figure.

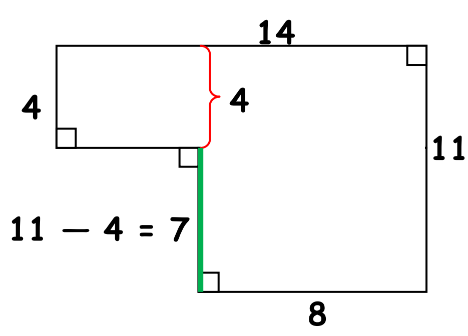
There is more than one way to break this figure into rectangles. We will only illustrate one below.





We have shown above that we can break the shape up into a red rectangle (figure on left) and a green rectangle (figure on right). We have the lengths of both sides of the red rectangle. It does not matter which one we call the base and which we call the height. The area of the red rectangle is $A = bh = 4 \times 14 = 56$

We have to do a little more work to find the area of the green rectangle. We know that the length of one of the sides is 8 units. We had to find the length of the other side of the green rectangle when we calculated the perimeter in Example 1 above. Its length was 7 units.



Thus the area of the green rectangle is $A = bh = 8 \times 7 = 56$. Thus the area of the whole figure is
area of red rectangle + area of green rectangle $= 56 + 56 = 112$. In

the process of calculating the area, we multiplied units times units. This will produce a final reading of square units (or units squared). Thus the area of the figure is 112 square units. This fits well with the definition of area which is the number of square units that will cover a closed figure.

Our next formula will be for the area of a parallelogram. A *parallelogram* is a quadrilateral with opposite sides parallel. parallelogram

Area of a Parallelogram

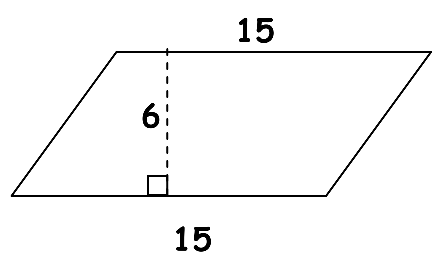
$$A = bh$$

***b* = the base of the parallelogram**
***h* = the height of the parallelogram**

You will notice that this is the same as the formula for the area of a rectangle. A rectangle is just a special type of parallelogram. The height of a parallelogram is a segment that connects the top of the parallelogram and the base of the parallelogram and is perpendicular to both the top and the base. In the case of a rectangle, this is the same as one of the sides of the rectangle that is perpendicular to the base.

Example 3:

Find the area of the figure below



Solution:

In this figure, the base of the parallelogram is 15 units and the height is 6 units. This mean that we only need to multiply to find the area of $A = bh = 15 \times 6 = 90$ square units.

You should notice that we cannot find the perimeter of this figure since we do not have the lengths of all of the sides, and we have no way to figure out the lengths of the other two sides that are not given.

Our next formula will be for the area of a trapezoid.

Area of a Trapezoid

$$A = \frac{1}{2}(b_1 + b_2)h$$

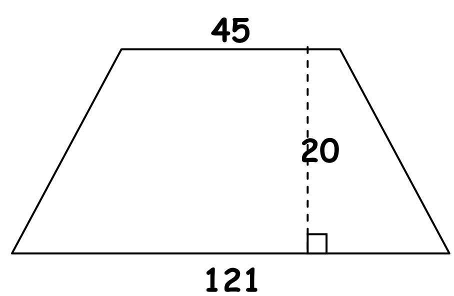
b_1 = the one base of the trapezoid
b_2 = the other base of the trapezoid
h = the height of the trapezoid

A *trapezoid* is a quadrilateral that has one pair of sides which are parallel. These two sides are called the bases of the trapezoid. The height of a trapezoid is a segment that connects the one base of the trapezoid and the other base of the trapezoid and is perpendicular to both of the bases.

trapezoid

Example 4:

Find the area of the figure



Solution:

For this trapezoid, the bases are shown as the top and the bottom of the figure. The lengths of these sides are 45 and 121 units. It does not matter which of these we say is b_1 and which is b_2. The height of the trapezoid is 20 units. When we plug all this into the formula, we get $A = \frac{1}{2}(b_1 + b_2)h = \frac{1}{2}(121 + 45)20 = 1660$ square units.

Our next formulas will be for finding the area of a *triangle* (a three-sided polygon). We will have more than one formula for this since there are different situations that can come up which will require different formulas triangle

Area of a Triangle

For a triangle with a base and height

$$A = \frac{1}{2}bh$$

***b* = the base of the triangle**
***h* = the height of the triangle**

Heron's Formula for a triangle with only sides

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

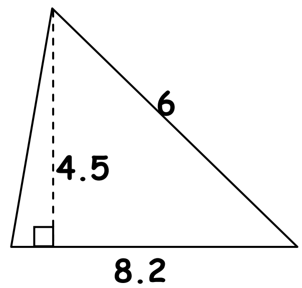
***a* = one side of the triangle**
***b* = another side of the triangle**
***c* = the third side of the triangle**
$s = \frac{1}{2}(a + b + c)$

The height of a triangle is the perpendicular distance from any vertex of a triangle to the side opposite that vertex. In other words the height of triangle is a segment that goes from the vertex of the triangle opposite the base to the base (or an extension of the base) that is perpendicular to the base (or an extension of the base). Notice that in this description of the height of a triangle, we had to include the words "or an extension of the base". This is required because the height of a triangle does not always fall within the sides of the triangle. Another thing to note is that any side of the triangle can be a base. You want to pick the base so that you will have the length of the base and also the length of the height to that base. The base does not need to be the bottom of the triangle.

You will notice that we can still find the area of a triangle if we don't have its height. This can be done in the case where we have the lengths of all the sides of the triangle. In this case, we would use Heron's formula.

Example 5:

Find the area of the figure.



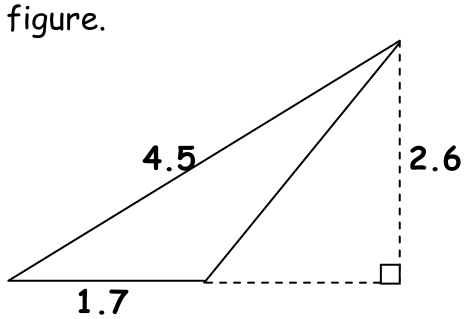
Solution:

Notice that in this figure has a dashed line that is shown to be perpendicular to the side that is 8.2 units in length. This is how we indicate the height of the triangle (the dashed line) and the base of the triangle (the side that the dashed line is perpendicular to). That means we have both the height and the base of this triangle, so we can just plug these numbers into the formula to get $A = \frac{1}{2}bh = \frac{1}{2}(8.2)(4.5) = 18.45$ square units.

Notice that the number 6 is given as the length of one of the sides of the triangle. This side is not a height of the triangle since it is not perpendicular to another side of the triangle. It is also not a base of the triangle, since there is no indication of the perpendicular distance between that side and the opposite vertex. This means that it is not used in the calculation of the area of the triangle.

Example 6:

Find the area of the figure.

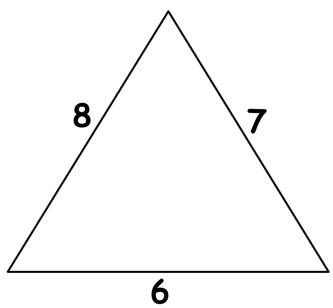


Solution:

In this figure there are two dashed lines. One of them is extended from the side of the triangle that has a length of 1.7. That dashed line is show to be perpendicular to the dashed line that has a length of 2.6. This is how we indicate that the dashed line that has a length of 2.6 is the height of the triangle and the base of the triangle is the side of length 1.7. The height here is outside of the triangle. Also, the dashed line that is extended from the base is not part of the triangle and its length is not relevant to finding the area of the triagnle. Since we have both the height and the base of this triangle, we can just plug these numbers into the formula to get $A = \frac{1}{2}bh = \frac{1}{2}(1.7)(2.6) = 2.21$ square units.

Example 7:

Find the area of the figure.



Solution:

You should notice that we do not have a height for this triangle. This means that we cannot use the formula that we have been using to find the area of this triangle. We do have the length of all three sides of the triangle. This means that we can use Heron's Formula to find the area of this triangle.

Heron's Formula

$A = \sqrt{s(s-a)(s-b)(s-c)}$

a = one side of the triangle
b = another side of the triangle
c = the third side of the triangle
$s = \frac{1}{2}(a + b + c)$

For this formula, it does not matter which side we label a, b, or c. For our purposes, we will let a be 6, b be 7, and c be 8. Now that we have a, b, and c, we need to calculate s so that we can plug a, b, c , and s into the formula. We get

$s = \frac{1}{2}(6+7+8) = \frac{1}{2}(21) = 10.5$. Now we can plug everything in to Heron's formula to find the area of this triangle to be
$A = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{10.5(10.5-6)(10.5-7)(10.5-8)} = 20.33315257$ square units.

Another formula that we are interested in is the Pythagorean Theorem. This applies to only right triangles. The Pythagorean Theorem relates the lengths of the sides of a right triangle.

Pythagorean Theorem

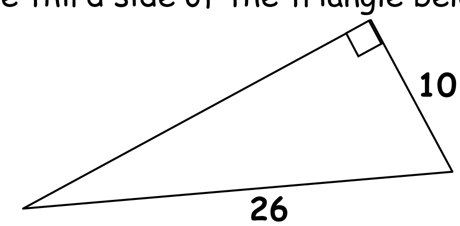
$$a^2 + b^2 = c^2$$

***a* = leg (one side of the triangle that makes up the right angle)**
***b* = leg (another side of the triangle that makes up the right angle)**
***c* = hypotenuse (side opposite the right angle)**

When using the Pythagorean Theorem, it is important to make sure that we always use the legs of the triangle for a and b and the hypotenuse for c.

Example 8:

Find the length of the third side of the triangle below.



Solution:

The figure is a right triangle (as indicated by the box in one of the angles of the triangle). We need to decide what the side we are looking for is in terms of a leg or the hypotenuse of the triangle. The hypotenuse is the side of the triangle opposite the right angle. That would be the side that has length 26 in our picture. Thus c will be 26 in our formula. This means that the other two sides of the triangle are legs *a* and *b*. We will let *a* be 10, and we will thus be looking for *b*. When we plug all this into the formula, we get

$$a^2 + b^2 = c^2$$
$$10^2 + b^2 = 26^2$$
$$100 + b^2 = 676$$

We now need to solve this equation for *b*.

$$100 + b^2 = 676$$
$$b^2 = 576$$
$$b = \sqrt{576} = 24$$

Since we are asked for a length, we have that the third side is 24 units. (Notice that this is not square units since we are not finding an area).

We are now going to move on to circles. We already mentioned the perimeter (circumference) of a circle in the perimeter sections. We also need a formula for finding the area of a circle.

Area of a Circle

$$A = \pi r^2$$

***r* = radius of the circle**
π = the number that is approximated by 3.141593

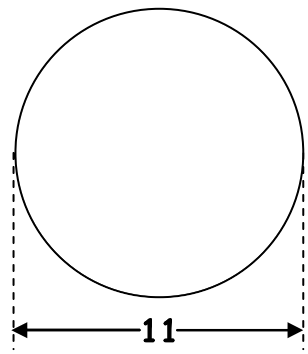
For circle problems we need to remember that the circumference (perimeter) of a circle is only the curved part of the circle. It does not include either the radius or diameter of the circle. In order to find the perimeter and area, we will need the radius of the circle. Recall that the *diameter* of a circle is a segment from on point on the circle to another point on the circle that passes through the center of the circle. A *radius* of a circle is half of a diameter (i.e. a segment that from one point on the circle to the center of the circle).

diameter

radius

Example 9:

Find the perimeter and area of the circle below.



Solution:

Circumference (Perimeter) of a Circle

$C = 2\pi r$

r = radius of the circle
π = the number that is approximated by 3.141593

The number 11 in the figure above is the length of the diameter of this circle. We need the radius to be able to use the formulas. The radius is half of the diameter. Thus $r = \frac{1}{2}(\text{diameter}) = \frac{1}{2}(11) = 5.5$.

We are now ready to find both the perimeter (circumference) and area by plugging 5.5 into each formula for r.

Perimeter:

$$C = 2\pi r = 2(\pi)(5.5) = 11\pi \approx 34.557519$$

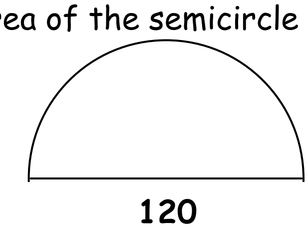
The answer 11π units is an exact answer for the perimeter. The answer 34.557519 units is an approximation.

Area:

$A = \pi r^2 = \pi(5.5)^2 = 30.25\pi \approx 95.0331778$ The answer 30.25π square units is an exact answer for the area. The answer 95.0331778 square units is an approximation.

Example 10:

Find the perimeter and area of the semicircle below.



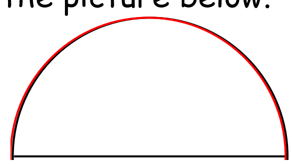
Solution:

In this problem, we have a semicircle (a half of a circle). The diameter of this half circle is 120. Thus the radius is $r = \frac{1}{2}(120) = 60$.

Perimeter:

We now can find the perimeter. We will find the circumference of the whole circle and then divide it by 2 since we only have half of a circle. This will give us $C = 2\pi(60) = 120\pi$ as the circumference for the whole circle and 60π units as the circumference of half of the circle.

Unfortunately this is not the answer for the perimeter of the figure given. The circumference of a circle is the curved part. The straight line segment in our figure would not have been part of the circumference of the whole circle, and thus it is not included as part of half of the circumference. We only have the red part shown in the picture below.



120

We still need to include the straight segment that is 120 units long. Thus the whole perimeter of the figure is curved part + straight part $= 60\pi + 120 \approx 308.495559$ units.

Area:

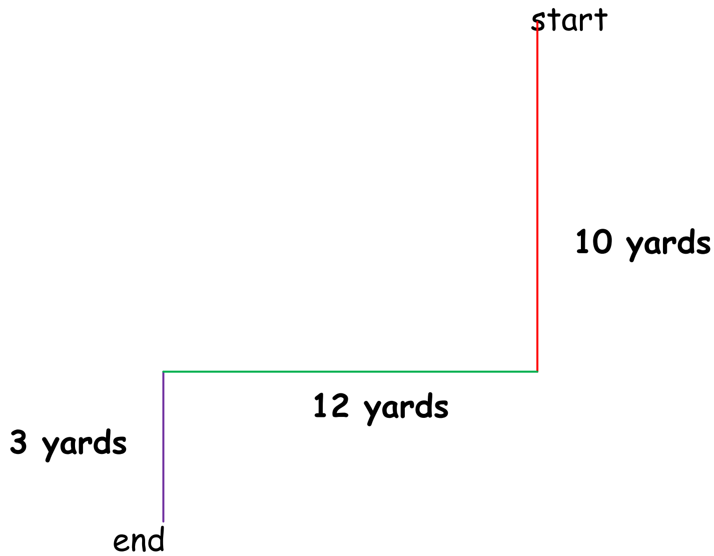
We can also find the area. Here we will plug 60 in for r in the area formula for a whole circle and then divide by 2 for the half circle. This will give us $A = \pi(60)^2 = 3600\pi$ square units for the area of the whole circle and 1800π square units for the area of the half circle. Now are we done with finding the area or is there more that we need to do like we did in finding the perimeter? If we think back to the definition of the area, (it is the number of square units needed to cover the figure) we should see that there is nothing further to do. The inside of the figure was covered already. We just have half as much inside as we did before.

Example 11:

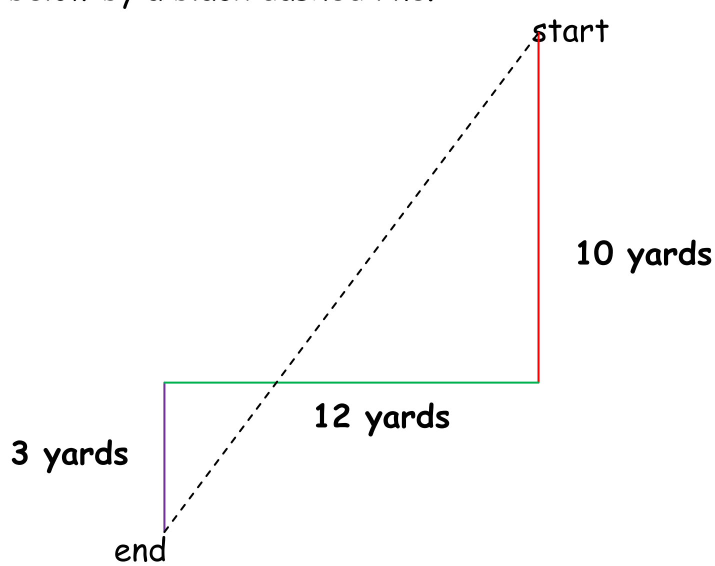
Walk 10 yards south, then 12 west and then 3 yards south. How far from the original starting point are you?

Solution:

To solve this problem, it would help to have a picture.



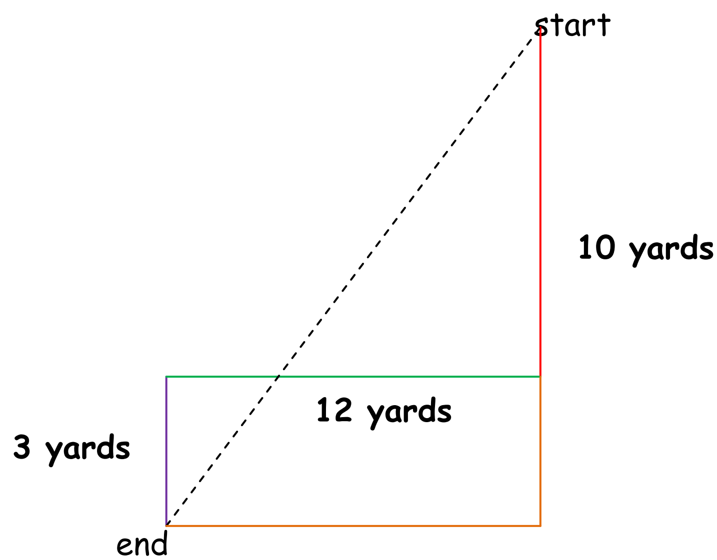
Above we can see a description of how we walked in the problem. What we are asked to find is how far it is from where we started to where we ended. That distance would be represented by a straight line from the start to the finish. This will be shown in the picture below by a black dashed line.



Geometry Notes

We need to figure out how long this line is. We might be tempted to say that we have two right triangles and that the line we are looking for is just the sum of each triangle's hypotenuse. The only problem with that is that we do not know the lengths of the legs of each of the right triangles. Thus we are not able to apply the Pythagorean Theorem in this way.

We are on the right track though. This is a Pythagorean Theorem problem. If we add a couple of line segments, we can create a right triangle whose hypotenuse is the line we are looking for.



As you can see above, the orange segments along with the red segment that was 10 yards and the dashed segment make up a right triangle whose hypotenuse is the dashed segment. We can figure out how long each of the legs of this triangle are fairly easily. The orange segment that is an extension of the red segment is another 3 yards. Thus the vertical leg of this triangle is 10 + 3 = 13 yards long. The orange segment which is horizontal is the same length as the green segment. This means that the horizontal leg of this triangle is 12 yards long. We can now use the Pythagorean Theorem to calculate how far away from the original starting point we are.

Pythagorean Theorem

$$a^2 + b^2 = c^2$$

a **= leg**
b **= leg**
c **= hypotenuse**

$$13^2 + 12^2 = c^2$$
$$169 + 144 = c^2$$
$$313 = c^2$$
$$\sqrt{313} = c \approx 17.6918$$

Here an exact answer does not really make sense. It is enough to approximate that we end up 17.6918 yards from where we started.

VOLUME AND SURFACE AREA

Objectives:

After completing this section, you should be able to do the following:

- Calculate the volume of given geometric figures.
- Calculate the surface area of given geometric figures.
- Solve word problems involving volume and surface area.

Vocabulary:

As you read, you should be looking for the following vocabulary words and their definitions:

- volume
- surface area
- sphere
- great circle of a sphere
- pyramid
- cone

Formulas:

You should be looking for the following formulas as you read:

- volume of a rectangle solid
- surface area of a rectangular solid
- volume of a cylinder
- surface area of a cylinder
- volume of a solid with a matching base and top
- volume of a sphere
- surface area of a sphere
- volume of a pyramid
- volume of a cone

We will continue our study of geometry by studying three-dimensional figures. We will look at the two-dimensional aspect of the outside covering of the figure and also look at the three-dimensional space that the figure encompasses.

The *surface area* of a figure is defined as the sum of the areas of the exposed sides of an object. A good way to think about this would be as the surface area

area of the paper that it would take to cover the outside of an object without any overlap. In most of our examples, the exposed sides of our objects will polygons whose areas we learned how to find in the previous section. When we talk about the surface area of a sphere, we will need a completely new formula.

The *volume* of an object is the amount of three-dimensional space an object takes up. It can be thought of as the number of cubes that are one unit by one unit by one unit that it takes to fill up an object. Hopefully this idea of cubes will help you remember that the units for volume are cubic units. volume

Surface Area of a Rectangular Solid (Box)

$$SA = 2(lw + lh + wh)$$

l **= length of the base of the solid**
w **= width of the base of the solid**
h **= height of the solid**

Volume of a Solid with a Matching Base and Top

$$V = Ah$$

A **= area of the base of the solid**
h **= height of the solid**

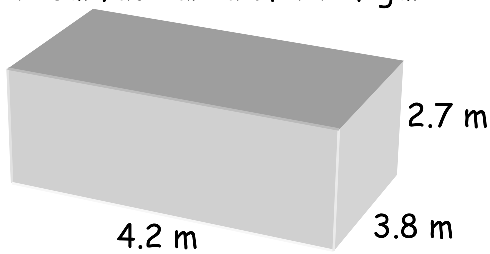
Volume of a Rectangular Solid
(specific type of solid with matching base and top)

$$V = lwh$$

l **= length of the base of the solid**
w **= width of the base of the solid**
h **= height of the solid**

Example 1:

Find the volume and the surface area of the figure below



Solution:

This figure is a box (officially called a rectangular prism). We are given the lengths of each of the length, width, and height of the box, thus we only need to plug into the formula. Based on the way our box is sitting, we can say that the length of the base is 4.2 m; the width of the base is 3.8 m; and the height of the solid is 2.7 m. Thus we can quickly find the volume of the box to be $V = lwh = (4.2)(3.8)(2.7) = 43.092$ cubic meters.

Although there is a formula that we can use to find the surface area of this box, you should notice that each of the six faces (outside surfaces) of the box is a rectangle. Thus, the surface area is the sum of the areas of each of these surfaces, and each of these areas is fairly straight-forward to calculate. We will use the formula in the problem. It will give us $SA = 2(lw + lh + wh) = 2(4.2 * 3.8 + 4.2 * 2.7 + 3.8 * 2.7) = 75.12$ square meters.

A *cylinder* is an object with straight sides and circular ends of the same size. The volume of a cylinder can be found in the same way you find the volume of a solid with a matching base and top. The surface area of a cylinder can be easily found when you realize that you have to find the area of the circular base and top and add that to the area of the sides. If you slice the side of the cylinder in a straight line from top to bottom and open it up, you will see that it makes a rectangle. The base of the rectangle is the circumference of the circular base, and the height of the rectangle is the height of the cylinder. cylinder

Volume of a Cylinder

$$V = Ah$$

A **= the area of the base of the cylinder**
h **= the height of the cylinder**

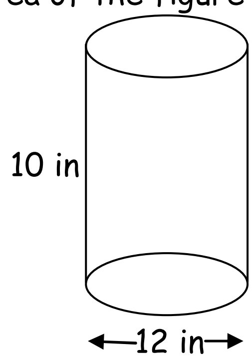
Surface Area of a Cylinder

$$SA = 2(\pi r^2) + 2\pi rh$$

r **= the radius of the circular base of the cylinder**
h **= the height of the cylinder**
π **= the number that is approximated by 3.141593**

Example 2:

Find the volume and surface area of the figure below



Solution:

This figure is a cylinder. The diameter of its circular base is 12 inches. This means that the radius of the circular base is $r = \frac{1}{2}d = \frac{1}{2}(12) = 6$ inches. The height of the cylinderi s 10 inches.

To calculate the volume and surface area, we simply need to plug into the formulas.

Surface Area:

$SA = 2(\pi r^2) + 2\pi rh = 2(\pi \cdot 6^2) + 2\pi(6)(10) = 72\pi + 120\pi = 192\pi$ square units. This is an exact answer. An approximate answer is 603.18579 square units.

Area of a Circle

$A = \pi r^2$

r = radius of the circle
π = the number that is approximated by 3.141593

Volume:
In order to plug into the formula, we need to recall how to find the area of a circle (the base of the cylinder is a circle). We will the replace A in the formula with the formula for the area of a circle.
$V = Ah = \pi r^2 h = \pi(6^2)(10) = 360\pi$ cubic inches. An approximation of this exact answer would be 1130.97336 cubic inches.

Our next set of formulas is going to be for spheres. A *sphere* is most easily thought of as a ball. The official definition of a sphere is a three-dimensional surface, all points of which are equidistant from a fixed point called the center of the sphere. A circle that runs along the surface of a sphere to that it cuts the sphere into two equal halves is called a *great circle of that sphere*. A great circle of a sphere would have a diameter that is equal to the diameter of the sphere.

sphere

great circle of a sphere

Surface Area of a Sphere

$$SA = 4\pi r^2$$

r = the radius of the sphere
π = the number that is approximated by 3.141593

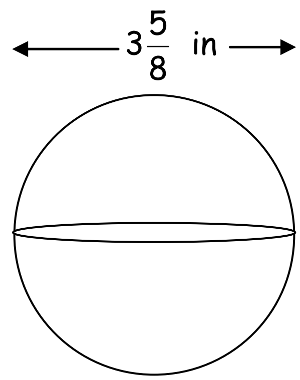
Volume of a Sphere

$$V = \frac{4}{3}\pi r^3$$

r = the radius of the sphere
π = the number that is approximated by 3.141593

Example 3:

Find the volume and surface area of the figure below



Solution:

This is a sphere. We are given that the diameter of the sphere is $3\frac{5}{8}$ inches. We need to calculate the radius of the sphere to calculate the volume and surface area. The radius of a sphere is half of its diameter. This means that the radius is $r = \frac{1}{2}d = \frac{1}{2}\left(3\frac{5}{8}\right) = \frac{1}{2}\left(\frac{29}{8}\right) = \frac{29}{16} = 1.8125$ inches. We can now just plug this number in to the formulas to calculate the volume and surface area.

Volume: $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\left(1.8125^3\right) \approx 24.941505$ cubic inches.

Surface Area: $SA = 4\pi r^2 = 4\pi\left(1.8125^2\right) \approx 41.28219$ square inches.

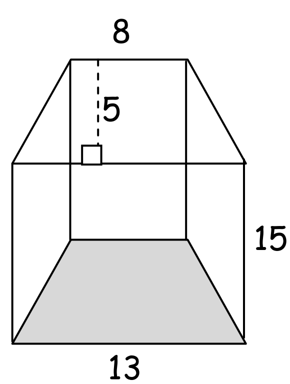
Example 4:

Find the volume of the figure

Volume of a Solid with a Matching Base and Top

$V = Ah$

***A*= area of the base of the solid**
***h* = height of the solid**



Solution:

> **Area of a Trapezoid**
>
> $$A = \frac{1}{2}(b_1 + b_2)h$$
>
> **b_1 = the one base of the trapezoid**
> **b_2 = the other base of the trapezoid**
> **h = the height of the trapezoid**

This figure is a solid with the same shape base and top. The shape of the base and top is a trapezoid. Thus we will need to remember the formula for the area of a trapezoid. For this trapezoid, the lengths of the bases are 13 and 8 units. It does not matter which of these we say is b_1 and which is b_2. The height of the trapezoid is 5 units. The height of the solid is 15 units. We will start by plugging the information about the trapezoidal base into the formula for the area of a trapezoid. Once we have this area, we will plug that and the height of the solid into the volume formula.

Area of the trapezoidal base:

$A = \frac{1}{2}(b_1 + b_2)h = \frac{1}{2}(13 + 8)5 = 52.5$ square units.

Volume of trapezoidal solid:

$V = Ah = (52.5)(15) = 787.5$ cubic units.

Our next formulas will be for finding the volume of a *cone* or a *pyramid*. These two formulas are grouped together since they are very similar. Each is basically 1/3 times the area of the base of the solid times the height of the solid. In the case of the cone, the base is a circle. In the case of the pyramid, we will have a base that is a rectangle. The height in both cases is the perpendicular distance from the apex to the plane which contains the base.

A pyramid is a solid figure with a polygonal base (in our case a rectangle) and triangular faces that meet at a common point (the apex). A cone is the surface of a conic solid whose base is a circle. This is more easily thought of as a pointed ice-cream cone whose top is circular and level. pyramid cone

> **Volume of a Rectangular Pyramid**
>
> $$V = \frac{1}{3}lwh$$
>
> **l = the length of the base of the pyramid**
> **w = the width of the base of the pyramid**
> **h = the perpendicular height of the pyramid**

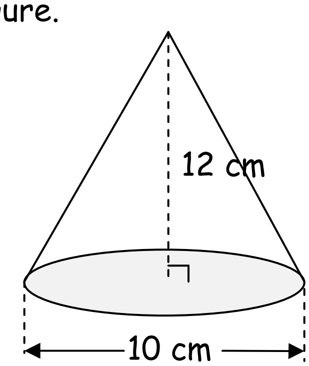
Volume of a Cone

$$V = \frac{1}{3}\pi r^2 h$$

***r* = the radius of the circular base of the cone**
***π* = the number that is approximated by 3.141593**
***h* = the perpendicular height of the cone**

Example 5:

Find the volume of the figure.

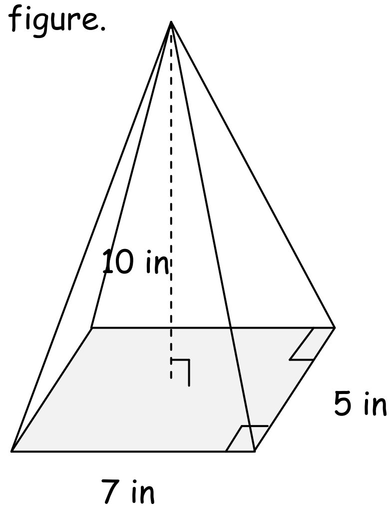


Solution:

Since the figure has a circular base and looks like an ice cream cone, this must be a cone. In order to find the volume of a cone, we need the radius of the circular base and the height (perpendicular height) of the cone. The height is given as 12 centimeters. The other measurement of 10 centimeters is the diameter of the circular base. We thus must calculate the radius to get $r = \frac{1}{2}d = \frac{1}{2}(10) = 5$ centimeters. We are now ready to plug into the volume of a cone formula. $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(5^2\right)(12) = 100\pi$ cubic centimeters is the exact volume. An approximation of this volume would be 314.159266 cubic centimeters.

Example 6:

Find the volume of the figure.



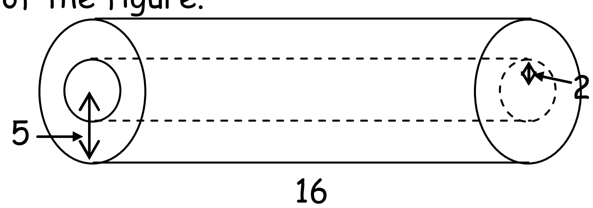
Solution:

The base of this figure is a rectangle and the sides of the figure are triangles, thus this figure is a rectangular pyramid. The height (perpendicular height) is 10 inches. The length of the base is 7 inches, and the width of the base is 5 inches. Since we have all of the parts for the volume formula, we can just plug into the volume of a rctangular pyramid formula to get $V = \frac{1}{3}lwh = \frac{1}{3}(7)(5)(10) = \frac{350}{3}$ cubic inches. An approximation of this volume would be 116.66667 cubic inches.

Just like with areas, we can add and subtract volumes of different solids to find the volume of a solid that is a combination of more than one solid or that have one solid removed from another.

Example 7:

Find the volume of the figure.



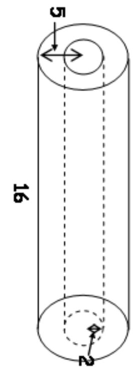
Solution:

Volume of a Cylinder

$V = Ah$

A **= the area of the base of the cylinder**
h **= the height of the cylinder**

The first thing that we need to do is figure out what type of figure this is. If we rotate the solid 90 degrees to the right, we get a figure that looks like this



This looks like a cylinder with the middle missing. A good way to think about this figure is a roll of paper towels. We are trying to find out the volume of the paper towels. The best way to do this is to figure out what the volume of the larger cylinder is without the missing part. We can then find the volume of the smaller cylinder or missing part. Finally, we will subtract the volume of the smaller cylinder from the volume of the larger cylinder to get the volume of our current solid.

We will start with the larger cylinder. The radius of the circular base is given as 5 units. The height of the larger cylinder is 16 units. We can then calculate the volume of the larger cylinder to be $V = Ah = \pi r^2 h = \pi(5^2)(16) = 400\pi$ cubic units.

Next we will calculate the volume of the smaller cylinder. The radius of the circular base of the smaller cylinder is 2 units. The height of the smaller cylinder is 16 units. We can calculate the volume of the smaller cylinder would be
$V = Ah = \pi r^2 h = \pi(2^2)(16) = 64\pi$ cubic units.

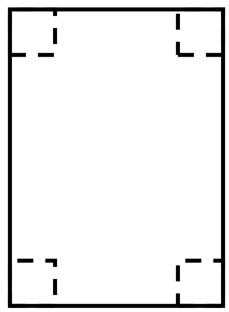
We now subtract the volume of the smaller cylinder from the volume of the larger cylinder to get the volume of our solid.
Volume of larger cylinder – volume of smaller cylinder =
$400\pi - 64\pi = 336\pi$ cubic units. This is approximately 1055.57513 cubic units.

Example 8:

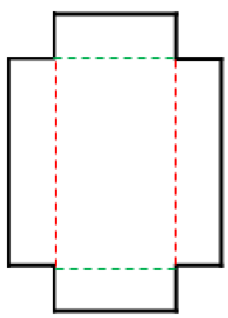
From an 8.5-inch by 11-inch piece of cardboard, 2-inch square corners are cut out and the resulting flaps are folded up to form an open box. Find the volume and surface area of the box.

Solution:

For this problem, it will be really helpful to make the box described above. You start with a standard piece of paper. You the cut out the dashed square indicated below from each corner. You make sure each side of the square is 2 inches in length.



What you are left with is the shape below.



You now fold up along the dashed lines to create a box. The box that we have created is a rectangular solid. This box has no top. Not having a top will not affect the volume of the box.

We only need to determine the length of the base of the box, the width of the base of the box, and the height of the box. The red dashed lines represent the length of the base of the box. The original length of the paper was 11 inches. We removed 2 inches from the top of the page and we also removed 2 inches from the bottom of the page. Thus the red dashed line is $11-2-2=7$ inches.

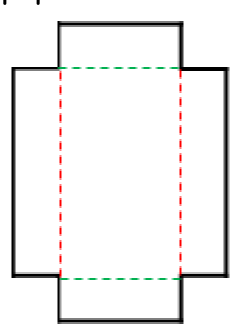
The green dashed line represents the width of the base of the box. The original width of the paper was 8.5 inches. We removed 2 inches from the left side of the page and also removed 2 inches from the right side of the page. Thus the green dashed line is $8.5-2-2=4.5$ inches.

We now need to think about the height of the box. Since we have folded up the sides for form the height of the box, we just need to determine how tall those sides are. Since they were made by cutting out 2-inch square from each corner, these sides must be 2 inches high.

Now we are ready of calculate the volumes
$V = lwh = (7)(4.5)(2) = 63$ cubic inches.

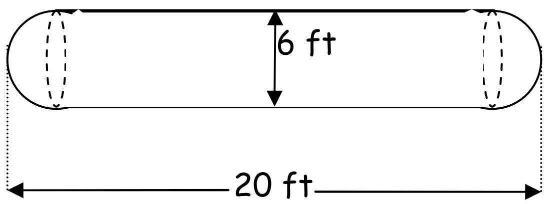
Now we need to calculate the surface area of our box. Since there is no top to this box, we can start formula for the surface area of a box. We will then need to subtract off the area of the top of the box. This will give us
$SA = 2(lw + wh + lh) = 2(7\cdot 4.5 + 4.5\cdot 2 + 7\cdot 2) = 109$ square inches for the box with the top included. The top would have the same area as the base of the box. This would be $A = lw = (7)(4.5) = 31.5$ square inches. Thus the surface area of our figure is total surface area – area of the top = $109-31.5 = 77.5$ square inches.

There is another way to calculate the surface area of this box. The surface area is the amount of paper it would take to cover the box without overlap. You should notice that this is the same as the amount of paper we used to make the box. Thus, it is enough to calculate the area of the paper as shown here.



Example 9:

A propane gas tank consists of a cylinder with a hemisphere at each end. Find the volume of the tank if the overall length is 20 feet and the diameter of the cylinder is 6 feet.



Solution:

We are told this tank consists of a cylinder (one its side) with a hemisphere at each end. A hemisphere is half of a sphere. To find the volume, we need to find the volume of the cylinder and the volumes of each hemisphere and then adding them together.

Let's start with the hemisphere sections. The diameter of the circular base of the cylinder is indicated to be 6 feet. This would also be the diameter of the hemispheres at each end of the cylinder. We need the radius of the sphere to find its volume. Once we calculate the volume of the whole sphere, we multiply it by $\frac{1}{2}$ to find the volume of the hemisphere (half of a sphere). We calculate the radius to be $r = \frac{1}{2}d = \frac{1}{2}(6) = 3$ feet. Now we can calculate the volume of the two hemispheres at the ends of the tank.

Left Hemisphere:

The volume of a whole sphere is $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\left(3^3\right) = 36\pi$ cubic feet. We now multiply this volume by $\frac{1}{2}$ to find the volume of the hemisphere to get $\frac{1}{2}V = \frac{1}{2}\left(36\pi\right) = 18\pi$ cubic feet.

> **Volume of a Sphere**
>
> $$V = \frac{4}{3}\pi r^3$$
>
> **r = the radius of the sphere**
> **π = the number that is approximated by 3.141593**

Right Hemisphere:

The volume of a whole sphere is $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\left(3^3\right) = 36\pi$ cubic feet. We now multiply this volume by ½ to find the volume of the hemisphere to get $\frac{1}{2}V = \frac{1}{2}\left(36\pi\right) = 18\pi$ cubic feet.

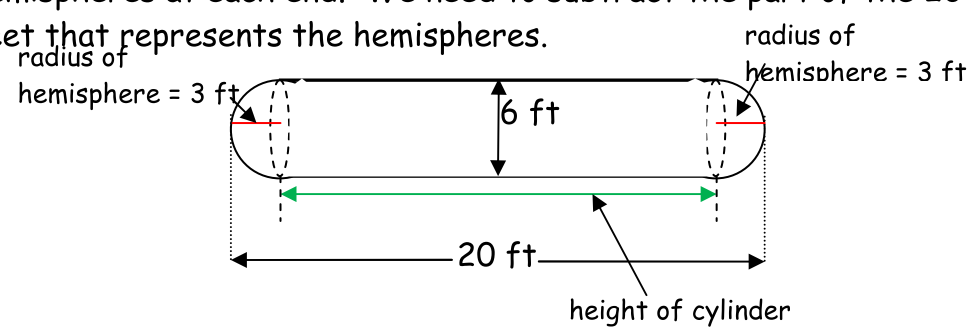
You might notice that we went through exactly the same process with exactly the same number for each hemisphere. We could have shortened this process by realizing that putting together the two hemispheres on each end, which were of the same diameter, would create a whole sphere. We could just have calculated the volume of this whole sphere.

Volume of a Cylinder

$V = Ah$

***A* = the area of the base of the cylinder**
***h* = the height of the cylinder**

Now we need to calculate the volume of the cylinder. We need the radius of the cylinder and the height of the cylinder to find its volume. The radius of the cylinder is the same as the radius of the hemispheres at each end. Thus the radius of the cylinder is 3 feet. It may look like the height of the cylinder is 20 feet. It turns out that this is not the case. The 20 feet includes the hemispheres at each end. We need to subtract the part of the 20 feet that represents the hemispheres.



Looking at the figure above, we see that the distance from the end of the left hemisphere to the left end of the cylinder is the radius of the hemisphere. The radius of the hemisphere is 3 feet. Similarly, the distance from the right end of the cylinder to the end of the right cylinder is also the radius of the hemisphere. The radius of the hemisphere is 3 feet. If we now subtract these two distances from the overall length of the tanks, we will have the height of the cylinder to be $h = 20 - 3 - 3 = 14$ feet. We can now calculate the volume of the cylinder.

Cylinder
$V = Ah = \pi r^2 h = \pi(3^2)(14) = 126\pi$ cubic feet.

We can now find the volume of the tank by adding together the volumes of the cylinder, the right hemisphere, and the left hemisphere. We get the volume to be $V = 126\pi + 18\pi + 18\pi = 162\pi$ cubic feet. This is approximated by 508.938 cubic feet.

Example 10:

A regulation baseball (hardball) has a great circle circumference of 9 inches; a regulation softball has a great circle circumference of 12 inches.

a. Find the volumes of the two types of balls.
b. Find the surface areas of the two types of balls.

Solution:

Part a:

In order to find the volume of a sphere, we need the radius of the sphere. In this problem, we are not given the radius. Instead we are given the circumference of a great circle of the sphere. From this information, we can calculate the radius of the great circle, which is the radius of the sphere.

Circumference (Perimeter) of a Circle

$C = 2\pi r$

***r* = radius of the circle**
***π* = the number**

Radius of the baseball:
We calculate the radius of the baseball by plugging in the circumference of the great circle of the baseball into the formula for the circumference of the circle and solve for r (the radius).

$$C = 2\pi r$$
$$9 = 2\pi r$$
$$\frac{9}{2\pi} = r$$

Volume of a Sphere

$$V = \frac{4}{3}\pi r^3$$

r = the radius of the sphere
π = the number that is approximated by 3.141593

Now that we have the radius of the baseball, we can calculate the volume, by plugging the radius into the formula for the volume of a sphere.

$$V = \frac{4}{3}\pi r^3$$
$$V = \frac{4}{3}\pi\left(\frac{9}{2\pi}\right)^3$$
$$V = \frac{4 \cdot 9^3 \pi}{3 \cdot 2^3 \pi^3}$$
$$V = \frac{2916\pi}{24\pi^3}$$
$$V = \frac{243}{2\pi^2} \text{ cubic inches}$$

This is approximately 12.3105 cubic inches.

Radius of the softball:
We will go through the same process with the softball to calculate the radius of the softball.

$$C = 2\pi r$$
$$12 = 2\pi r$$
$$\frac{12}{2\pi} = r$$
$$\frac{6}{\pi} = r$$

Now that we have the radius of the softball, we can calculate the volume, by plugging the radius into the formula for the volume of a sphere.

$$V = \frac{4}{3}\pi r^3$$
$$V = \frac{4}{3}\pi\left(\frac{6}{\pi}\right)^3$$
$$V = \frac{4 \cdot 6^3 \pi}{3 \cdot \pi^3}$$
$$V = \frac{864\pi}{3\pi^3}$$
$$V = \frac{288}{\pi^2} \text{ cubic inches}$$

This is approximately 29.1805 cubic inches.

Surface Area of a Sphere

$SA = 4\pi r^2$

***r* = the radius of the sphere**
π = the number that is approximated by 3.141593

Part b:

In order to find the surface area of a sphere, we need the radius of the sphere. We calculated the radius for each type of ball in part a. We only need to plug this information in to the formula for surface area of a sphere.

Surface area of a baseball:

$$SA = 4\pi\left(\frac{9}{2\pi}\right)^2$$

$$SA = \frac{4\pi(9)^2}{2^2\pi^2}$$

$$SA = \frac{324\pi}{4\pi^2}$$

$$SA = \frac{81}{\pi} \text{ square inches}$$

This is approximately 25.7831 square inches.

Surface area of a softball:

$$SA = 4\pi\left(\frac{6}{\pi}\right)^2$$

$$SA = \frac{4\pi(6)^2}{\pi^2}$$

$$SA = \frac{144\pi}{\pi^2}$$

$$SA = \frac{144}{\pi} \text{ square inches}$$

This is approximately 45.83662 square inches.

Our final example is an application problem. We will need to be able to use dimensional analysis, volumes, and common sense in order to be able to answer the question.

Example 11:

Mike Jones bought an older house and wants to put in a new concrete driveway. The driveway will be 30 feet long, 10 feet wide, and 9 inches thick. Concrete (a mixture of sand, gravel, and cement) is measured by the cubic yard. One sack of dry cement mix costs $7.30, and it takes four sacks to mix up 1 cubic yard of concrete. How much will it cost Mike to buy the cement?

Solution:

The driveway that is being poured will be a rectangular solid or box. Thus in order to answer this question, we will first need to find the volume of this box (the amount of cubic units it will take to fill this box). The problem tells us that concrete is measured by the cubic yard. This lets us know that those are the units we will want to calculate with. None of the dimensions of the driveway are given in yards. We will need to use our dimensional analysis (unit conversion) to convert all of the measurements to yards. This could be done at a later point, but this is the easiest place to take care of the conversion.

Convert length:

$$\frac{30\text{ feet}}{1}\cdot\frac{1\text{ yard}}{3\text{ feet}}=10\text{ yards}$$

Convert width:

$\dfrac{10\text{ feet}}{1}\cdot\dfrac{1\text{ yard}}{3\text{ feet}}=\dfrac{10}{3}$ yards We are not going to estimate this value since the approximation with introduce error. As in the finance section, we don't want to round until the end of the problem or in a place where it is absolutely necessary.

Convert height:

$\dfrac{9\text{ inches}}{1}\cdot\dfrac{1\text{ yard}}{36\text{ inches}}=\dfrac{1}{4}\text{ yards}=.25\text{ yards}$ Since the decimal representation of this number is a terminating decimal, we can use this representation in our calculations.

Volume of driveway:

We are now ready to calculate the volume of the driveway.

$V=lwh=(10)\left(\dfrac{10}{3}\right)(.25)=\dfrac{25}{3}$ cubic yards.

Now, we are told that it takes 4 bags of cement to make one cubic yard of concrete. So we will now calculate how many bags of cement to buy. Mike's driveway is $\frac{25}{3}$ cubic yards and each of those cubic yards requires 4 bags of cement. Thus Mike will need $\frac{25}{3} \cdot 4 = \frac{100}{3}$ bags or approximately 33.333 bags. Here is where common sense needs to come in. There is no store that will sell .333 bags of cement mix. Stores only sell whole bags of cement mix. Thus we will need to round up to the next whole bag (we can't round down or we will not have enough cement mix to complete the driveway). This means that Mike will need to buy 34 bags of cement mix. Each of these bags will cost \$7.30. This means that Mike will pay $7.30 \cdot 34 = \$248.20$ for the cement mix for this driveway.

SIMILAR TRIANGLES

Objectives:

After completing this section, you should be able to do the following:

- Calculate the lengths of sides of similar triangles.
- Solve word problems involving similar triangles.

Vocabulary:

As you read, you should be looking for the following vocabulary words and their definitions:

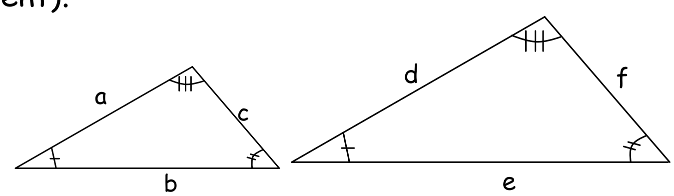
- similar triangles

Formulas:

You should be looking for the following formulas as you read:

- proportions for similar triangles

We will continue our study of geometry by looking at similar triangles. Two triangles are *similar* if their corresponding angles are congruent (or have the same measurement). similar



In the picture above, the corresponding angles are indicated in the two triangles by the same number of hash marks. In other words, the angle with one hash mark in the smaller triangle corresponds to the angle with one hash mark in the larger triangle. These angles have the same measure or are congruent.

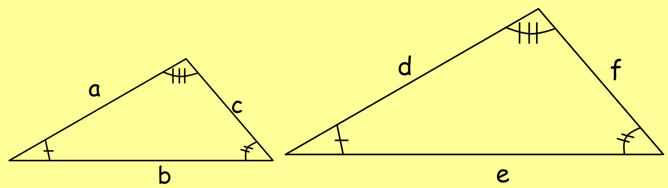
There are also corresponding sides in similar triangles. In the triangles above side *a* corresponds to side *d* (for example). These sides do not necessarily have the same measure. They do, however, form a ratio that is the same no matter which pair of corresponding sides the ratio is made from. Thus we can write the following equation

$$\frac{a}{d} = \frac{b}{e} = \frac{c}{f}$$

Notice that the sides of one particular triangle are always written on top of the fractions and the sides of the other triangle are always written on the bottom of the fractions. It does not matter which triangle is put in which part of the fraction as longs as we are consistent within a problem.

Similar Triangle Ratios

$$\frac{a}{d} = \frac{b}{e} = \frac{c}{f}$$

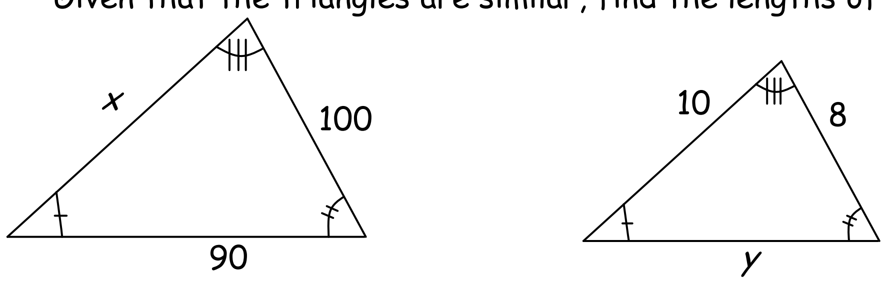


It should be noted that although our triangles are in the same relative position, this is not needed for triangles to be similar. One of the triangles can be rotated or reflected.

Please note that pictures below are not drawn to scale.

Example 1:

Given that the triangles are similar, find the lengths of the missing sides.



Solution:

There is one side missing in the triangle on the left. This side is labeled x. There is also one side missing from the triangle on the right. This side is labeled y. The side x in the triangle on the left corresponds to the side labeled 10 in the triangle on the right. We know this because these sides connect the angle with one hash mark to the angle with three hash marks in each of the triangles. The side 100 in the triangle on the left corresponds to the side labeled 8 in the triangle on the right. Finally the side 90 in the triangle on the left corresponds to the side labeled y in the triangle on the right. We will need this information to find values for x and y.

Find x:
We will start by finding the value for x. We will need to form a ratio with the side labeled x and is corresponding side in the triangle on the right. We will also need a pair of corresponding side both of whose measurements we have and make a ratio out of those. Once we make an equation using these ratios, we will just need to solve the equation for x. I am going to choose to place the sides of the triangle on the left on the top of each of the ratios.

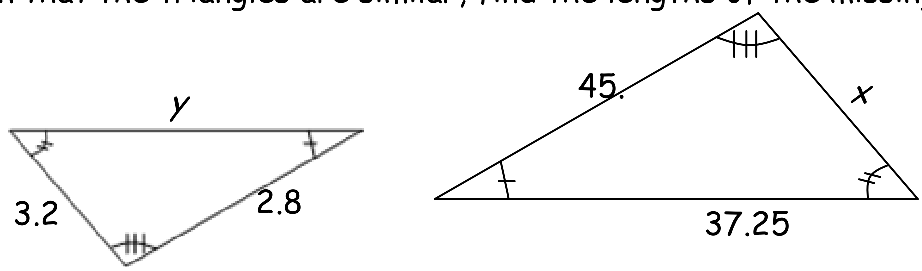
$$\frac{x}{10} = \frac{100}{8}$$
$$8(x) = 10(100)$$
$$8x = 1000$$
$$x = 125 \text{ units}$$

Find y:
We will continue by finding the value for y. We will need to form a ratio with the side labeled y and is corresponding side in the triangle on the right. We will also need a pair of corresponding side both of whose measurements we have and make a ratio out of those. Although we know the measurements of all the other sides of the triangle, it is best to avoid using a side we just calculate to make further calculations if possible. The reason for this is that if we have made a mistake in our previous calculation, we would end up with an incorrect answer here as well. Once we make an equation using these ratios, we will just need to solve the equation for y. I am going to choose to place the sides of the triangle on the left on the top of each of the ratios.

$$\frac{90}{y} = \frac{100}{8}$$
$$8(90) = y(100)$$
$$720 = 100y$$
$$7.2 \text{ units} = y$$

Example 2:

Given that the triangles are similar, find the lengths of the missing sides.



Solution:

Our first job here is to determine which sides in the triangle on the left correspond to which sides in the triangle on the right. The side in the triangle on the left labeled y joins the angle with one hash mark and the angle with two has marks. The side in the triangle on the right that does this is the side labeled 37.25. Thus these are corresponding sides.

In a similar manner, we can see that the side labeled 3.2 in the triangle on the left corresponds to the side labeled x on the triangle on the right. Finally the side labeled 2.8 in the triangle on the left corresponds to the side labeled 45 in the triangle on the right.

Find x:

We will start by finding the value for x. We will need to form a ratio with the side labeled x and is corresponding side in the triangle on the right. We will also need a pair of corresponding side both of whose measurements we have and make a ratio out of those. Once we make an equation using these ratios, we will just need to solve the equation for x. I am going to choose to place the sides of the triangle on the left on the top of each of the ratios.

$$\frac{3.2}{x} = \frac{2.8}{45}$$
$$3.2(45) = x(2.8)$$
$$144 = 2.8x$$
$$\frac{144}{2.8}\text{ units } = x$$

This is approximately 51.42857 units.

Find y:
We will continue by finding the value for y. We will need to form a ratio with the side labeled y and is corresponding side in the triangle on the right. We will also need a pair of corresponding side both of whose measurements we have and make a ratio out of those. Once we make an equation using these ratios, we will just need to solve the equation for y. I am going to choose to place the sides of the triangle on the left on the top of each of the ratios.

$$\frac{y}{37.25} = \frac{2.8}{45}$$
$$y(45) = 37.25(2.8)$$
$$45y = 104.3$$
$$y = \frac{104.3}{45}\text{ units}$$

This is approximately 3.21778 units.

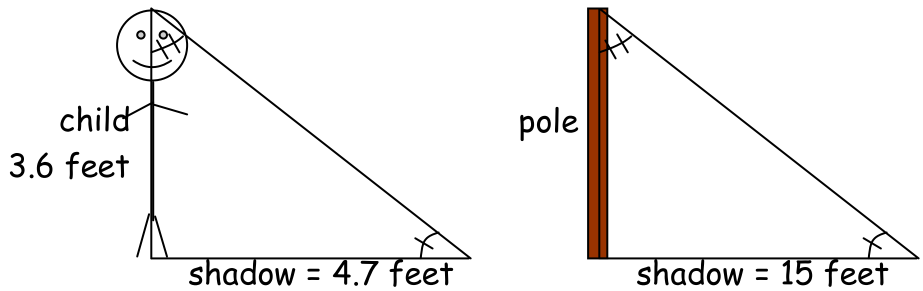
Our final example is an application of similar triangles

Example 3:

A 3.6-foot –tall child casts a shadow of 4.7 feet at the same instant that a telephone pole casts a shadow of 15 feet. How tall is the telephone pole?

Solution:

For this problem, it helps to have a picture.



When we look at the picture above, we see that we have two triangles that are similar. We are looking for the height of the pole. The pole side of the triangle on the right corresponds to the side of the child side of the triangle on the left. The two shadow sides of the triangles also correspond to each other. Thus we can write an equation of ratios to find the height of the pole.

$$\frac{3.6}{\text{pole}} = \frac{4.7}{15}$$

$$(3.6)(15) = \text{pole}(4.7)$$

$$54 = 4.7\text{pole}$$

$$\frac{54}{4.7} = \text{pole}$$

Thus the pole is approximately 11.48936 feet high.

RIGHT TRIANGLE TRIGONOMETRY

Objectives:

After completing this section, you should be able to do the following:

- Calculate the lengths of sides and angles of a right triangle using trigonometric ratios.
- Solve word problems involving right triangles and trigonometric ratios.

Vocabulary:

As you read, you should be looking for the following vocabulary words and their definitions:

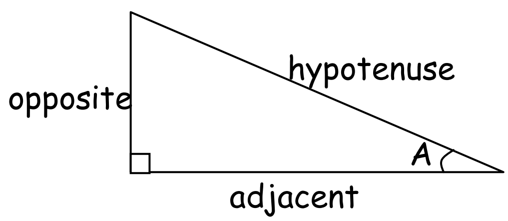
- hypotenuse
- adjacent side
- opposite side

Formulas:

You should be looking for the following formulas as you read:

- ratio for sin(*A*) where *A* is a non-right angle in right triangle
- ratio for cos(*A*) where *A* is a non-right angle in right triangle
- ratio for tan(*A*) where *A* is a non-right angle in right triangle

We will complete our study with a further study of right triangles. We will look at trigonometric value as defined by ratios of the sides of a right triangle.



In the figure above, you can see the sides of a right triangle labeled. The side labeled *hypotenuse* is always opposite the right angle of the right triangle. The names of the other two sides of the right triangle are determined by the angle that is being discusses. In our case, we will be discusing the sides in terms of the angle labeled *A*. The angle *A* is form by the hypotenuse of the right triangle and the side of the right triangle that

hypotenuse

is called *adjacent*. The *adjacent* side will always make up part of the angle that is being discussed and not be the hypotenuse. The side of the right triangle that does not form part of angle *A* is called the *opposite* side. The *opposite* side will never form part of the angle being discussed. adjacent opposite

The trigonometric function values of a particular value can be as the ratio of a particular pair of sides of a right triangle containing an angle of that measure. We will look at three particular trigonometric ratios.

Trigonometric Ratios

$$\sin(A) = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos(A) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

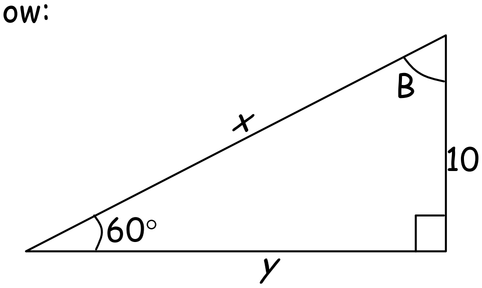
$$\tan(A) = \frac{\text{opposite}}{\text{adjacent}}$$

sin = shortened form of sine function
cos = shortened form of cosine function
tan = shortened form of tangent function
***A* = the angle value**

We will use these ratios to answer questions about triangles below and then we will go through a couple of application problems.

Example 1:

Use Trigonometric ratios to find the unknown sides and angles in the right triangles below:



Solution:

We are being asked to find values for x, y, and B. We will do the angle B first.

Angle B:

We can find the measure of angle B without using any trigonometric ratios. What we need to remember to find this value is that the sum of the three angles of a triangle will always add up to 180 degrees. It does not matter the size or shape of the triangle. The sum of the three angles will always be 180 degrees. We know that one angle is a right angle. Its measure is 90 degrees. The measure of the other angle is given to be 60 degrees. Thus we just need to calculate

$$90 + 60 + B = 180$$
$$150 + B = 180$$
$$B = 30$$

Thus the measure of angle B is 30 degrees.

Side x:

We will now work to find the length of side x. We need to start by determining which angle we are going to use for our problem. As in the past, it is best to use an angle that is given. Thus we will be using the angle labeled 60 degrees. Our next step is to determine which side x is relative to the angle labeled 60 degrees. Since the side labeled x is opposite the right angle, it is the hypotenuse. There are two trigonometric ratios that include the hypotenuse. Thus, we need to determine which one to use. This will be determined by the other side of the triangle whose measure we know. This is the side labeled 10. Since this side is not one of the sides of the triangle that makes up the angle labeled 60 degrees, it is the opposite side. This means that we need to use the trigonometric ratio that has both the hypotenuse and the opposite side. That ratio is the sine ratio. We will plug into that equation and solve for x.

Geometry Notes

$$\sin(A) = \frac{\text{opposite}}{\text{hypotenuse}}$$
$$\sin(60) = \frac{10}{x}$$
$$\frac{\sin(60)}{1} = \frac{10}{x}$$
$$x\sin(60) = 1(10)$$
$$x = \frac{10}{\sin(60)}$$

Through out this solution, we have left sin(60) in this form. This saves us from needing to round until the end of the problem. sin(60) is just a number that at the end of the problem can be calculated by our calculator. Our answer is approximately x = 11.54701.

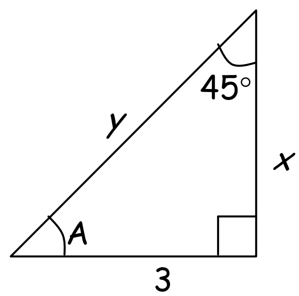
Side y:
We will finish by finding the length of side y. As in the previous par of the problem, it is best to use the angle labeled 60 degrees. Our next step is to determine which side y is relative to the angle labeled 60 degrees. Since the side labeled y is forms part of the angle labeled 60 degrees, it is the adjacent side. There are two trigonometric ratios that include the adjacent side. Since the other side that is given is the side labeled 10 and this side is the opposite side (see explanation above), we will need to use the trigonometric ratio that has both the opposite side and the adjacent side. That ratio is the tangent ratio. We will plug into that equation and solve for y.

$$\tan(A) = \frac{\text{opposite}}{\text{adjacent}}$$
$$\tan(60) = \frac{10}{y}$$
$$\frac{\tan(60)}{1} = \frac{10}{y}$$
$$y\tan(60) = 1(10)$$
$$y = \frac{10}{\tan(60)}$$

Through out this solution, we have left tan(60) in this form. This saves us from needing to round until the end of the problem. tan(60) is just a number that at the end of the problem can be calculated by our calculator. Our answer is approximately y = 5.773503.

Example 2:

Use Trigonometric ratios to find the unknown sides and angles in the right triangles below:



Solution:

As we solve this problem, we will leave out the explanations of how we determine the names of the sides of the triangle.

Angle A:

As with the previous problem, the sum of the angles of a triangle is 180 degrees. Thus we calculate

$$90 + 45 + A = 180$$
$$135 + A = 180$$
$$A = 45$$

Thus the measure of angle A is 45 degrees.

Side x:

Side x forms part of the angle that is labeled to be 45 degrees, thus this is the adjacent side. We are also given the measure of the side opposite the angle to be 3. Thus we will want to use the tangent ratio.

$$\tan(45) = \frac{3}{x}$$
$$\frac{\tan(45)}{1} = \frac{3}{x}$$
$$x\tan(45) = 1(3)$$
$$x = \frac{3}{\tan(45)}$$

Our answer is $x = 3$.

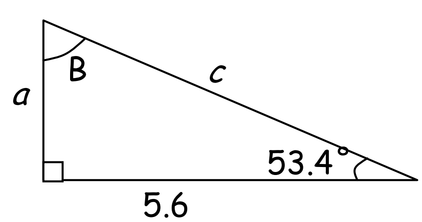
We did not need to use trigonometric ratios to find x. We could have used the fact that our triangle has two angles that are equal. Such a triangle is an isosceles triangle. We should recall that the sides of an isosceles triangle opposite the equal angles are equal in length. Thus since one of the side was length 3, the side labeled x is also of length 3.

Side y:
Side y is opposite the right angle of the triangle and thus is the hypotenuse. We also have given opposite side to be 3. Thus to find y, we will need to use the sine ratio.

$$\sin(45) = \frac{3}{y}$$
$$\frac{\sin(45)}{1} = \frac{3}{y}$$
$$y\sin(45) = 1(3)$$
$$y = \frac{3}{\sin(45)}$$

Our answer is approximately $y = 4.24264$.

Example 3:
Use Trigonometric ratios to find the unknown sides and angles in the right triangles below:



Solution:

Angle B:

As with the previous problem, the sum of the angles of a triangle is 180 degrees. Thus we calculate

$$90 + 53.4 + B = 180$$
$$143.4 + B = 180$$
$$B = 36.6$$

Thus the measure of angle B is 36.6 degrees.

Side c:

We will use the given angle labeled 53.4 degrees as our angle for solving this problem. Side c forms part of the triangle that is opposite the right angle. Thus it is the hypotenuse. We are also given the measure of the side adjacent to the angle we are using to be 5.6. Thus we will want to use the cosine ratio.

$$\cos(53.4) = \frac{5.6}{c}$$
$$\frac{\cos(53.4)}{1} = \frac{5.6}{c}$$
$$c \cdot \cos(53.4) = 1(5.6)$$
$$c = \frac{5.6}{\cos(53.4)}$$

Our answer is approximately c = 9.39243.

Side a:

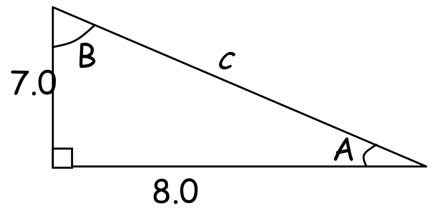
We will use the given angle labeled 53.4 degrees as our angle for solving this problem. Side a does not form the angle we are using. Thus it is the opposite side. We are also given the measure of the side adjacent to the angle we are using to be 5.6. Thus we will want to use the tangent ratio.

$$\tan(53.4) = \frac{a}{5.6}$$
$$\frac{\tan(53.4)}{1} = \frac{a}{5.6}$$
$$5.6 \cdot \tan(53.4) = 1(a)$$
$$5.6 \cdot \tan(53.4) = a$$

Our answer is approximately a = 7.54041.

Example 4:

Use Trigonometric ratios to find the unknown sides and angles in the right triangles below:



Solution:

In this problem, we are not given any angle to use. Instead we will need to change the labels of our sides as we solve each of the angles in turn. We will start by finding side c. Since this is a right triangle, we can use the Pythagorean theorem to find the length of c.

Side c:

The legs (a and b) are given to be 7.0 and 8.0. It does not matter which we label a and which we label b.

$$a^2 + b^2 = c^2$$
$$7.0^2 + 8.0^2 = c^2$$
$$49 + 64 = c^2$$
$$113 = c^2$$
$$\sqrt{113} = c$$

The approximate length of side c is 10.63015.

Angle A:

As we solve for angle A, we need to label the sides whose measures are given relative to angle A. The side labeled 8.0 forms part of the angle A. Thus it is the adjacent side. The side labeled 7.0 does not form any part of the angle A. Thus it is the opposite side. The trigonometric ratio that includes both the adjacent and opposite sides is the tangent ratio. We will fill in the information to that equation and solve for A.

$$\tan(A) = \frac{\text{opposite}}{\text{adjacent}}$$
$$\tan(A) = \frac{7.0}{8.0}$$
$$\tan(A) = 0.875$$

We need to know how to solve for *A* in this equation. As in our section on exponential functions and their inverses, there is an inverse function (a functions that undoes) for the tangent function. On the calculator it is labeled $\tan^{-1}$. Thus we can finally solve for *A* by calculating

$$\tan(A) = 0.875$$
$$A = \tan^{-1}(0.875)$$

Thus the measure of angle *A* is approximately *A* = 41.18593 degrees.

Angle *B*:
As we solve for angle *B*, we need to relabel the sides whose measures are given relative to angle *B*. The side labeled 7.0 forms part of the angle *B*. Thus it is the adjacent side. The side labeled 8.0 does not form any part of the angle *B*. Thus it is the opposite side. The trigonometric ratio that includes both the adjacent and opposite sides is the tangent ratio. We will fill in the information to that equation and solve for *A*.

$$\tan(A) = \frac{\text{opposite}}{\text{adjacent}}$$
$$\tan(A) = \frac{8.0}{7.0}$$

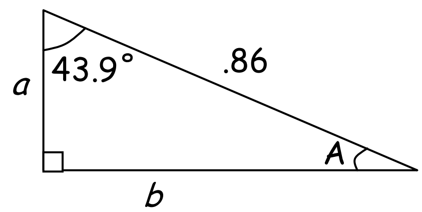
We will once again use the inverse function of the tangent function. On the calculator it is labeled $\tan^{-1}$. Thus we can finally solve for *B* by calculating

$$\tan(B) = \frac{8.0}{7.0}$$
$$B = \tan^{-1}\left(\frac{8.0}{7.0}\right)$$

Thus the measure of angle *B* is approximately *B* = 48.81407 degrees.

Example 5:

Use Trigonometric ratios to find the unknown sides and angles in the right triangles below:



Solution:

Angle A:

As with the previous problem, the sum of the angles of a triangle is 180 degrees. Thus we calculate

$$90 + 43.9 + A = 180$$
$$133.9 + A = 180$$
$$A = 46.1$$

Thus the measure of angle A is 46.1 degrees.

Side a:

We will use the given angle labeled 43.9 degrees as our angle for solving this problem. Side a forms part of the angle labeled 43.9 degrees. Thus it is the adjacent side. We are also given the measure of the hypotenuse to be .86. Thus we will want to use the cosine ratio.

$$\cos(43.9) = \frac{a}{.86}$$
$$\frac{\cos(43.9)}{1} = \frac{a}{.86}$$
$$.86\cos(43.9) = 1(a)$$
$$.86\cos(43.9) = a$$

Our answer is approximately $a = 0.619674$.

Side b:

We will use the given angle labeled 43.9 degrees as our angle for solving this problem. Side b does not form the angle we are using. Thus it is the opposite side. We are also given the measure of the hypotenuse to be .86. Thus we will want to use the sine ratio.

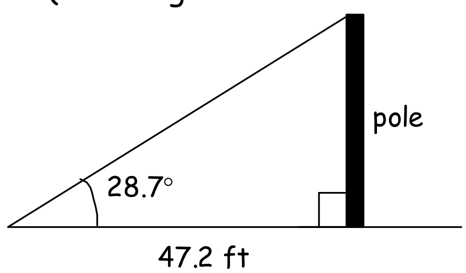
$$\sin(43.9) = \frac{b}{.86}$$
$$\frac{\sin(43.9)}{1} = \frac{b}{.86}$$
$$.86 \cdot \sin(43.9) = 1(b)$$
$$.86 \cdot \sin(43.9) = b$$

Our answer is approximately b = 0.596326.

We will finish by looking at some application problems for our right triangle trigonometric ratios.

Example 6:

A support cable runs from the top of the telephone pole to a point on the ground 47.2 feet from its base. If the cable makes an angle of 28.7° with the ground, find (rounding to the nearest tenth of a foot)



a. the height of the pole
b. the length of the cable

Solution:

The picture above shows that we have a right triangle situation.

Part a:

The pole is opposite the angle of 28.7 degrees that is given. The other side of the triangle that we know to be 47.2 feet forms part of the angle of 28.7 degrees. Thus it is the adjacent side. We will be able to use the tangent ratio to solve this problem since it includes both the opposite and adjacent sides.

$$\tan(A) = \frac{\text{opposite}}{\text{adjacent}}$$
$$\tan(28.7) = \frac{\text{pole}}{47.2}$$
$$\frac{\tan(28.7)}{1} = \frac{\text{pole}}{47.2}$$
$$47.2\tan(28.7) = 1(\text{pole})$$
$$47.2\tan(28.7) = \text{pole}$$

Thus the pole is approximately 25.8 feet tall.

Part b:

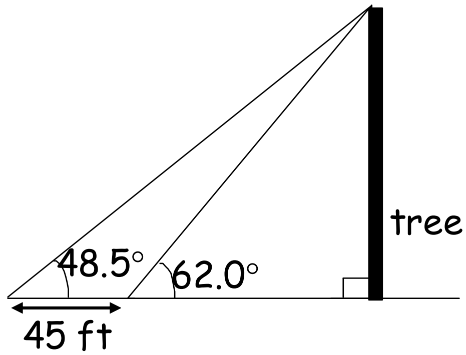
The cable is opposite the right angle of triangle and thus is the hypotenuse. We still know that the adjacent side is 47.2 feet. We will be able to use the cosine ratio to solve this problem since it includes both the hypotenuse and adjacent sides.

$$\cos(A) = \frac{\text{adjacent}}{\text{hypotenuse}}$$
$$\cos(28.7) = \frac{47.2}{\text{cable}}$$
$$\frac{\cos(28.7)}{1} = \frac{47.2}{\text{cable}}$$
$$\text{cable} \cdot \cos(28.7) = 1(47.2)$$
$$\text{cable} = \frac{47.2}{\cos(28.7)}$$

Thus the cable is approximately 53.8 feet tall.

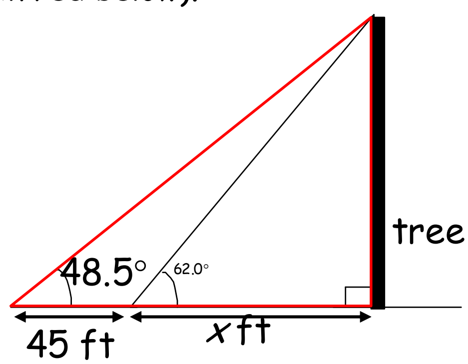
Example 7:

You are hiking along a river and see a tall tree on the opposite bank. You measure the angle of elevation of the top of the tree and find it to be 62.0°. You then walk 45 feet directly away from the tree and measure the angle of elevation. If the second measurement is 48.5°, how tall is the tree? Round your answer to the nearest foot.



Solution:

This problem will require a little more algebra than the previous problems. We will start by looking at the two different triangles that we have and writing trigonometric ratios that include the tree for each of our triangles. We will start by looking at the bigger triangles (shown in red below).

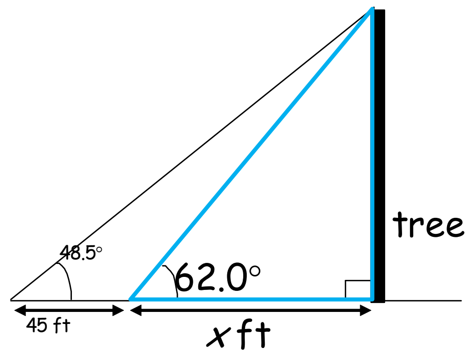


In this red triangle, the tree is opposite the angle that is given to be 48.5 degrees. We are also given the length of 45 feet as part of the side that is adjacent to the angle given to be 48.5 degrees. Since we have part of this adjacent side, we are going to label the other part of the side to be x feet. Thus the whole adjacent side is $45+x$ feet long. Since we have the opposite and the adjacent sides, we can use the tangent ratio. For the red triangle, we have

$$\tan(x) = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan(48.5) = \frac{\text{tree}}{45+x}$$

We will continue by looking at the smaller triangle (shown in blue below)



In this blue triangle, the tree is opposite the angle that is given to be 62 degrees. Based on what we did for the red triangle, we know that the length of the side adjacent to the angle given to be 62 degrees is x feet long. Since we have the opposite and the adjacent sides, we can use the tangent ratio. For the blue triangle, we have

$$\tan(x) = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan(62) = \frac{\text{tree}}{x}$$

Now we are ready for the algebra. We have two equations $\tan(48.5) = \frac{\text{tree}}{45 + x}$ and $\tan(62) = \frac{\text{tree}}{x}$ with two unknowns (variable). We can use the substitution method (solve one of the equations for one of the variable and then plug that in to the other equation) to determine the height of the tree. One way to go here is to solve the equation $\tan(62) = \frac{\text{tree}}{x}$ for x. This will give us

$$\tan(62) = \frac{\text{tree}}{x}$$

$$\frac{\tan(62)}{1} = \frac{\text{tree}}{x}$$

$$x\tan(62) = 1(\text{tree})$$

$$x = \frac{\text{tree}}{\tan(62)}$$

We can now plug this expression for x into the equation $\tan(48.5) = \frac{\text{tree}}{45 + x}$ and solve for the height of the tree.

$$\tan(48.5) = \frac{\text{tree}}{45 + x}$$

$$\frac{\tan(48.5)}{1} = \frac{\text{tree}}{45 + x}$$ Write both sides as fractions.

$$(45 + x)\tan(48.5) = \text{tree}$$ Cross multiply.

$$45\tan(48.5) + x\tan(48.5) = \text{tree}$$ Apply distributive property.

$$45\tan(48.5) + \left(\frac{\text{tree}}{\tan(62)}\right)\tan(48.5) = \text{tree}$$ Substitute in expression for *x*.

$$45\tan(48.5) = \text{tree} - \left(\frac{\text{tree}}{\tan(62)}\right)\tan(48.5)$$ Collect variable on the same side

$$45\tan(48.5) = \text{tree}\left(1 - \left(\frac{1}{\tan(62)}\right)\tan(48.5)\right)$$ Factor out the variable *tree*

$$\frac{45\tan(48.5)}{\left(1 - \left(\frac{1}{\tan(62)}\right)\tan(48.5)\right)} = \text{tree}$$ Divide both sides by value in parentheses.

We now just need to plug this expression into our calculator to find out the height of the tree. The tree is approximately 127 feet tall.